Indian Statistical Institute, Bangalore

B. Math.

First Year, Second Semester

Linear Algebra II

Final Examination — Instructor: B V Rajarama Bhat

Total Marks 7x15=105 — Date : April 22, 2026

Maximum marks: 100 — Time: 3 hours

Here $M_n(\mathbb{C})$ denotes the vector space of $n \times n$ complex matrices. The standard inner product is considered unless specified otherwise.

(1) Let $A = [a_{ij}]_{1 \le i,j \le n} \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and $\det(A) = 5$. Compute the determinant of $B = [b_{ij}]_{1 \le i,j \le 2n} \in M_{2n}(\mathbb{C})$ defined by

$$b_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i,j \le n; \\ a_{(i-n)(j-n)} & \text{if } n < i,j \le 2n; \\ 0 & \text{Otherwise.} \end{cases}$$

Justify your answer. [15]

(2) Let $C = [c_{ij}]_{1 \le i,j \le n} \in M_n(\mathbb{C})$ be a positive matrix. Take

$$M = \max\{|c_{ij}| : 1 \le i,j \le n\};$$

$$m = \min\{|c_{ij}| : 1 \le i,j \le n\}.$$

(i) Show that there exists $k \in \{1, 2, \ldots, n\}$ such that $c_{kk} = M$.

(ii) For $1 \le k,l \le m$, if $c_{kk} = m = c_{ll}$, show that $|c_{kl}| = |c_{lk}| = m$.

[15]

(3) Obtain an ortho-normal basis for $\mathbb{C}^3$, by Gram-Schmidt process applied to x_1, x_2, x_3, where

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(4) Let $A \in M_n(\mathbb{C})$. Show that there exists a real number s such that $sI + A$ is positive if and only if A is self-adjoint. [15]

(5) Obtain the polar and singular value decompositions for the following matrix:

$$N = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

[15]

(6) Show that two normal matrices are unitarily equivalent if and only if they are similar. Show that a non-normal matrix can be similar to a normal matrix without being unitarily equivalent. [15]

(7) Let J be the 3×3 Jordan block, whose diagonal entries are equal to d, for some $d \in \mathbb{C}$:

$$J = \begin{pmatrix} d & 1 & 0 \\ 0 & d & 1 \\ 0 & 0 & d \end{pmatrix}$$

Find the minimal polynomial and the characteristic polynomial of J^2. (Caution: Treat $d = 0$ carefully.) [15]